

Non-Associative Geometry and the Spectral Action Principle

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ABSTRACT: Chamseddine and Connes have shown how the action for Einstein gravity, coupled to the $SU(3) \times SU(2) \times U(1)$ standard model of particle physics, may be elegantly recast as the “spectral action” on a certain “non-commutative geometry.” In this paper, we show how this formalism may be extended to “non-associative geometries,” and explain the motivations for doing so. As a guiding illustration, we present the simplest non-associative geometry (based on the octonions) and evaluate its spectral action: it describes Einstein gravity coupled to a G_2 gauge theory, with 8 Dirac fermions (which transform as a singlet and a septuplet under G_2). We use this example to illustrate how non-associative geometries may be naturally linked to ordinary (associative) geometries by a certain twisting procedure. This is just the simplest example: in a forthcoming paper we show how to construct realistic models that include Higgs fields, spontaneous symmetry breaking and fermion masses.

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1. Introduction

Chamseddine and Connes have shown how the action for Einstein gravity, coupled to the $SU(3) \times SU(2) \times U(1)$ standard model of particle physics, may be elegantly recast as the “spectral action” on a certain “non-commutative geometry” [1–13]. In this paper, we show how and why this formalism should be extended to “non-associative geometries,” and evaluate the spectral action for the simplest example of a model based on a nonassociative geometry.

Here, in Section 1, we briefly present our motivations and summarize our ideas and results. In the first two Subsections, 1.1 and 1.2, we introduce the reader to “non-commutative” geometry (in the sense of Connes) and the spectral action principle,

respectively. These two subsections are meant to provide some orientation and motivation for readers who are not experts in these topics; they are not intended to be rigorous or complete, and can probably be skipped by experts. (For more in depth pedagogical introductions, aimed at physicists, see *e.g.* [14, 15].) In Subsection 1.3, we explain our motivations for extending this formalism from non-commutative to non-associative geometry. In Subsection 1.4 we summarize the contents of this paper, including the main ideas and results.

1.1 Noncommutative geometry a la Connes: What and Why?

Over the past few decades, the mathematician Alain Connes and others have pioneered an area of mathematics known as spectral (or noncommutative) geometry [2, 3, 16]. Much as Riemannian differential geometry was a generalization of Euclidean geometry, spectral geometry is a generalization of Riemannian geometry. Riemann extended the framework of geometry to include spaces that are curved, and spectral geometry further extends it to include spaces that are “noncommutative” in a sense we explain below.

In Riemann’s approach, one specifies a geometry by providing the following data: the manifold \mathcal{M} and its metric $g_{\mu\nu}$. In the spectral approach, one instead specifies a geometry by providing a so-called “spectral triple” $\{\mathcal{A}, \mathcal{H}, D\}$: here \mathcal{A} is an algebra, which is represented by linear operators acting on the Hilbert space \mathcal{H} , and D is an additional Hermitian operator on \mathcal{H} . In fact, to *fully* specify a spectral geometry, one often needs to give two additional operators, called γ and J , so that the full spectral data is $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$. Following convention, we will still call $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$ a “spectral triple,” even though it contains 5 elements! (The various elements of the spectral triple, and their meaning, are explained below.)

Riemannian geometry contains Euclidean geometry, and reduces to Euclidean geometry for a special class of Riemannian data: namely, when the manifold \mathcal{M} is given by \mathbb{R}^n and the metric $g_{\mu\nu}$ is given by the flat Euclidean metric $\delta_{\mu\nu}$. Similarly, spectral geometry contains Riemannian geometry, and reduces to Riemannian geometry for a special class of spectral data: namely, when the spectral triple $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$ is given by the so-called “canonical spectral triple” $M = \{\mathcal{A}_c, \mathcal{H}_c, D_c, \gamma_c, J_c\}$ (explained below). The idea is that the Riemannian data $\{\mathcal{M}, g_{\mu\nu}\}$ and the canonical triple M provide dual descriptions of the same geometry, so that the canonical spectral triple may be obtained from the Riemannian data, or vice versa.

Starting from the Riemannian data $\{\mathcal{M}, g_{\mu\nu}\}$, the corresponding canonical spectral triple M may be constructed as follows: $\mathcal{A}_c = C_\infty(\mathcal{M})$ is the algebra of smooth complex-valued functions on \mathcal{M} ; $\mathcal{H}_c = L^2(\mathcal{M}, S)$ is the Hilbert space of (square integrable) Dirac spinors on $\{\mathcal{M}, g_{\mu\nu}\}$; $D_c = -i\gamma^\mu \nabla_\mu^S$ is the ordinary curved-space Dirac operator on $\{\mathcal{M}, g_{\mu\nu}\}$; γ_c is the helicity operator on \mathcal{H}_c (*i.e.* what physicists usually

call γ_5 in 4 dimensions); and J_c is the charge conjugation operator on \mathcal{H}_c . As for the representation of \mathcal{A}_c on \mathcal{H}_c , the functions $f \in \mathcal{A}_c$ act on the spinor fields $\psi \in \mathcal{H}_c$ by pointwise multiplication: $\psi(x) \rightarrow f(x)\psi(x)$.

Conversely, starting from the canonical spectral triple M , one can reconstruct the Riemannian data $\{\mathcal{M}, g_{\mu\nu}\}$: for details, see [2, 3, 16, 17] and references therein. In essence, specifying the algebra \mathcal{A}_c amounts to specifying the manifold \mathcal{M} , while specifying the operator D_c (and, in particular, its spectrum of eigenvalues) amounts to specifying the metric $g_{\mu\nu}$. Let us try to unpack these statements a bit. How does the algebra \mathcal{A}_c encode the manifold \mathcal{M} ? According to the commutative Gelfand-Naimark theorem, any commutative algebra (or, more correctly, any commutative C^* -algebra) is equivalent to the algebra of complex-valued functions over a certain topological space $X_{\mathcal{A}}$ encoded by \mathcal{A} , where $X_{\mathcal{A}}$ is the space of “characters” of \mathcal{A} – *i.e.* $*$ -homomorphisms $\phi : \mathcal{A} \rightarrow \mathbb{C}$. In particular, for the algebra $\mathcal{A}_c = C_{\infty}(\mathcal{M})$, the characters (which are in one-to-one correspondence with the points $p \in \mathcal{M}$) are precisely the maps $\phi_p : \mathcal{A} \rightarrow \mathbb{C}$ given by $\phi_p(f) = f(p)$; and $\text{Diff}(\mathcal{M})$ (the group of diffeomorphisms of the manifold \mathcal{M}) is nothing but $\text{Aut}(\mathcal{A}_c)$ (the group of automorphism of the algebra \mathcal{A}_c). Next let us see how D_c (a differential operator) interacts with \mathcal{A}_c (the algebra of functions on \mathcal{M}) to encode the metric $g_{\mu\nu}$. To see how differentiation of functions on a manifold may be translated into distances on that manifold, it is enough to consider a simple 1-dimensional example: we can re-express the distance $|x_a - x_b|$ between two points x_a and x_b on the real line in a dual fashion, as the maximum possible excursion $|f(x_a) - f(x_b)|$ that any function $f(x)$ can make between those two points, subject to the constraint that its derivative cannot be too large ($|df/dx| \leq 1$). This example is just meant to convey the key idea: for more details, we again recommend that the reader consult [2, 3, 16, 17]. Let us add that the other elements of the canonical spectral triple also have geometric meaning: the operator γ_c , which provides a notion of orientation (left or right handedness) of spinors, encodes the volume form of the underlying Riemannian geometry; and the anti-unitary operator J_c , which maps spinors to their anti-spinors, provides a “real structure” to the geometry, much as the operation of complex conjugation selects a preferred line (the real line) inside the complex plane.

Why would one trade the familiar Riemannian data $\{\mathcal{M}, g_{\mu\nu}\}$ for the unfamiliar spectral data M ? From the purely mathematical standpoint, one of the key reasons is that, once one has reformulated geometry in spectral terms, one finds that it naturally extends to a class of spectral triples that is far broader than the subclass of canonical spectral triples described above. Thus one obtains a natural generalization of Riemannian geometry, capable of handling many mathematical spaces that lie beyond the boundaries of ordinary Riemannian geometry. In particular, although the algebra \mathcal{A}_c in the canonical triple is commutative, the algebra \mathcal{A} in the general triple need not be;

for this reason, spectral geometry is also called noncommutative geometry.

So much for the mathematical meaning and significance of spectral geometry; let us now explain why it is of interest from the standpoint of physics.

1.2 The Spectral Action: What and Why?

In spectral geometry, there is a natural action functional called the “spectral action,” which assigns a real number to each spectral triple. It is given by the simple formula

$$S = \text{Tr}[f(D/\Lambda)] + \langle \psi | D | \psi \rangle \quad (1.1)$$

where $f(x)$ is a real, even function of a single variable, which vanishes rapidly for $|x| > 1$; and ψ is an element of the Hilbert space \mathcal{H} . This general expression may be derived from the “spectral action principle” (*i.e.* the requirement the action should only depend on the spectrum of the operator D), together with the requirement that the action of the union of two geometries is the sum of their respective actions, as usual.

Let us first apply the action (1.1) to the canonical spectral triple M . Note that we are imagining evaluating the spectral action on a Euclidean manifold. The lorentzian case is then obtained by analytic continuation. In Eq. (1.1), the $\text{Tr}[f(D_c/\Lambda)]$ term may be expanded in powers of Λ , using the standard heat kernel expansion; the leading terms in this expansion are given by

$$\text{Tr}[f(D_c/\Lambda)] = \int d^4x \sqrt{g} \left(\frac{f_4 \Lambda^4}{2\pi^2} - \frac{f_2 \Lambda^2}{24\pi^2} R + \dots \right) \quad (1.2)$$

where $f_n \equiv \int_0^\infty f(x) x^{n-1} dx$. In other words, this term reduces to the ordinary action for Einstein gravity (cosmological constant term plus Einstein-Hilbert term), so that the term $\text{Tr}[f(D/\Lambda)]$ may be regarded as the natural generalization of Einstein gravity in the context of spectral geometry. Meanwhile, the second term in (1.1) becomes:

$$\langle \psi | D_c | \psi \rangle = \int d^4x \sqrt{g} \psi^\dagger(x) D_c \psi(x) = -i \int d^4x \sqrt{g} \psi^\dagger \gamma^\mu \nabla_\mu^S \psi(x). \quad (1.3)$$

In other words, this term reduces to the Euclidean action for a single, massless Dirac fermion.

Now let us look at what happens when we apply the spectral action (1.1) to a spectral triple that is *non*-canonical (and, in particular, noncommutative). The key result is that, for a certain (rather simple and natural) class of spectral triples $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$, the spectral action (1.1) reduces precisely to the action for Einstein gravity coupled to the full $SU(3) \times SU(2) \times U(1)$ standard model of particle physics, in all its detail. In particular, $\text{Tr}[f(D/\Lambda)]$ produces the bosonic terms in the action (the gravitational

terms, the kinetic terms for the gauge bosons, and the kinetic and potential terms for the Higgs doublet), while $\langle\psi|D|\psi\rangle$ produces the fermionic terms (the kinetic terms for the leptons and quarks, their Yukawa interactions with the Higgs doublet, and the neutrino mass terms). For the detailed derivation of this result, see [1–13, 15].

It is worth pausing to emphasize the difference between this perspective on noncommutativity, and the one more commonly encountered in the physics literature. Often, noncommutative geometry is taken to mean the noncommutativity of the 4-dimensional spacetime coordinates themselves; it is regarded as a property of quantum gravity that presumably becomes manifest at the Planck energy scale (*i.e.* the exceedingly high energy scale of 10^{19} GeV). By contrast, from the perspective of the spectral reformulation of the standard model, all of the non-gravitational fields in nature at low energies are reinterpreted as the direct manifestations of noncommutative geometry, right in front of our nose, staring us in the face!

What is the motivation for reformulating the familiar action for the standard model (coupled to gravity) in the unfamiliar language of spectral triples and spectral action? We would like to stress four points:

1) Simplicity. The spectral action (1.1) packages all of the complexity of gravity and the standard model of particle physics into two simple and elegant terms which, in turn, follow from a simple principle (the spectral action principle described above). The compactness and tautness of this formulation suggest that it may be a step in the right direction. To give a provocative analogy: much as Minkowski “discovered” that the rather cumbersome Lorentz transformations (which formed the basis of Einstein’s original formulation of special relativity) could be elegantly re-interpreted as the geometrical statement that we live in a 4-dimensional Minkowski spacetime, Chamseddine and Connes seem to have discovered that the rather cumbersome action for the standard model coupled to gravity can be elegantly re-interpreted as the geometric statement that we live in a certain type of noncommutative geometry. (Einstein initially rejected Minkowski’s unfamiliar formulation of special relativity as worse than useless, calling it “superfluous learnedness,” and quipping that “since the mathematicians have tackled the theory of relativity, I myself no longer understand it anymore” [18]. Ultimately, of course, it proved to be a crucial step on the road to general relativity.)

2) Symmetry. As we generalize from Riemannian geometry to spectral geometry, the algebra \mathcal{A} generalizes the manifold \mathcal{M} ; and the group $\text{Aut}(\mathcal{A})$ of automorphisms of \mathcal{A} generalizes the group $\text{Diff}(\mathcal{M})$ of diffeomorphisms of \mathcal{M} . In particular, in the canonical spectral triple, where $\mathcal{A} = C_\infty(\mathcal{M}, \mathbb{C})$ is the commutative $*$ -algebra of smooth functions $f : \mathcal{M} \rightarrow \mathbb{C}$, we have $\text{Aut}(\mathcal{A}) = \text{Diff}(\mathcal{M})$. Now consider the next simplest case, in which $\mathcal{A} = C_\infty(\mathcal{M}, M_n(\mathbb{C}))$ is the noncommutative $*$ -algebra of smooth functions $f : \mathcal{M} \rightarrow M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the set of $n \times n$ complex matrices. In

this case, $\text{Aut}(\mathcal{A})$ is the semi-direct product of two groups

$$\text{Aut}(\mathcal{A}) = \text{Map}(\mathcal{M}, SU(N)) \ltimes \text{Diff}(\mathcal{M}) \quad (1.4)$$

where $\text{Map}(\mathcal{M}, SU(N))$ is the group of maps from \mathcal{M} to the group $SU(N)$. Notice that the group on the right-hand side of Eq. (1.4) also has another interpretation: it is the full symmetry group of $SU(N)$ gauge theory coupled to Einstein gravity – namely, the semi-direct product of the group $\text{Map}(\mathcal{M}, SU(N))$ of gauge transformations and the group $\text{Diff}(\mathcal{M})$ of gravitational symmetries. Indeed, if one evaluates the spectral action (1.1) for a spectral triple based on this algebra $C_\infty(\mathcal{M}, M_n(\mathbb{C}))$, one finds that it reduces to $SU(N)$ gauge theory coupled to Einstein gravity [4]. In this example, we see that an elegant and conceptually satisfying picture emerges: the full symmetry group of a gauge theory coupled to gravity is reinterpreted in a unified way as simply $\text{Aut}(\mathcal{A})$, the automorphism group of an underlying algebra; and this, in turn, is interpreted as the group of “purely gravitational” transformations of a corresponding noncommutative space. In essence, this basic picture is also behind the spectral reformulation of the standard model coupled to gravity.

To see what is compelling about this picture, let us contrast it with Kaluza-Klein (KK) theory. To see the contrast clearly, it is enough to consider the original and simplest KK model. In this model, one starts with the 5D Einstein-Hilbert action $S_{KK} = (16\pi G_5)^{-1} \int d^5X \sqrt{-g_5} R_5$, where X , G_5 , g_5 and R_5 denote the 5D spacetime coordinates, Newton constant, metric determinant and Ricci scalar, respectively. Next, one supposes that the 5D manifold is the product of a 4D manifold and a circle ($\mathcal{M}_5 = \mathcal{M}_4 \times S_1$), where x^μ are the 4D coordinates on \mathcal{M}_4 and z is the coordinate on S_1 . Finally, one writes the general 5D line element in the form

$$ds_5^2 = g_{mn}^{(5)} dX^m dX^n = e^{\varphi/\sqrt{3}} g_{\mu\nu}^{(4)} dx^\mu dx^\nu + e^{-2\varphi/\sqrt{3}} (A_\mu dx^\mu + dz)^2 \quad (1.5)$$

and observes that, if the 5D metric $g_{mn}^{(5)}$ only depends on the 4D coordinates x^μ , then the 5D action S_{KK} reduces to a 4D action of the form

$$S_{KK} = \int d^4x \sqrt{-g_4} \left[\frac{R_4}{16\pi G_4} - \frac{1}{2} (\partial\varphi)^2 - \frac{1}{4} e^{-\sqrt{3}\varphi} F^2 \right] \quad (1.6)$$

where G_4 , g_4 and R_4 are the 4D Newton constant, metric determinant and Ricci scalar, respectively, while $F_{\mu\nu}$ is the 4D Maxwell field strength derived from the 4D gauge potential A_μ . This simple example captures what is appealing about KK theory – that one starts from the simple and purely gravitational action for Einstein gravity in 5D and obtains something tantalizingly close to 4D Einstein gravity plus 4D gauge theory – but it also captures what is unappealing about KK theory. For one thing,

one typically obtains extra, unwanted fields with unwanted couplings (in this example, the massless scalar field φ , with its experimentally untenable $e^{-\sqrt{3}\varphi}F^2$ coupling to electromagnetism), and one must explain why these extra fields and couplings are not observed in nature. For another thing, the reduction from the initial 5D action, which has the huge symmetry group $\text{Diff}(\mathcal{M}_5)$, to the final 4D action, which has the much smaller symmetry group $\text{Map}(\mathcal{M}_4, U(1)) \ltimes \text{Diff}(\mathcal{M}_4)$, fundamentally relies on the assumption that the 5D metric $g_{mn}^{(5)}$ only depends on the 4D coordinates x^μ . This assumption is supposed to be justified, in turn, by the fact that the compactified direction is so small; but this justification assumes that one has stabilized the extra dimension – *i.e.* found a way to make it small and keep it small, without letting it shrink down to a singularity or blow up to macroscopic size. The problem of stabilizing extra dimensions in KK theory is a famously thorny one and, furthermore, is ultimately at the root of the so-called landscape problem in string theory.

Thus the spectral and KK approaches share a similar spirit: in both cases the goal is to reinterpret the action describing ordinary 4-dimensional physics as arising from a simpler action formulated on an “extension” of 4-dimensional spacetime. But the spectral action seems to achieve this goal more elegantly and directly. In KK theory, the starting point is an action with too many fields and too much symmetry, and one must then jump through many hoops to explain why these extra fields and symmetries are unobserved in nature. By contrast, in the spectral approach, the field content and symmetries of the standard model are obtained directly.

3) Unification: In the 5D KK model, we see that 3 different objects [(i) the 4D metric, (ii) the 4D gauge field and (iii) the 4D dilaton field] are unified, in the sense that they are all packaged together as different parts of the metric on the extended (5D) geometry. Analogously, in the spectral action, four different objects [(i) the 4D Levi-Civita connection (which is related to the 4D metric), (ii) the 4D gauge fields, (iii) the Higgs field and (iv) the matrix of Yukawa couplings] are unified, in the sense that they are all packaged together as different parts of the Dirac operator D_* on the extended (spectral) geometry. Note that in the spectral case, the 4 objects which are unified in this way are all crucial and experimentally verified components of the standard model of particle physics. In the spectral framework, the remaining fields (*i.e.* the fermions) are nothing but the basis vectors on the Hilbert space \mathcal{H} .

4) Higgs structure. In ordinary gauge theory, we start by choosing a gauge group; and after we do this, the number of gauge bosons, and their transformation properties are fixed (*i.e.* they are “output”), while the number of scalar (Higgs) bosons and their properties can still be chosen by hand (*i.e.* they are additional “input”). This is to be contrasted with the situation in spectral geometry: here the gauge bosons and scalar Higgs bosons are on the same footing, and are interpreted geometrically as two different

aspects of the Dirac operator on a spectral geometry. In particular, their number and properties of the gauge bosons and Higgs bosons are both *output* – *i.e.* they are both computed as part of the same computation. In particular, in the spectral reformulation of the standard model of particle physics, the Higgs boson is not put in by hand (as it is in the usual approach); rather, its existence and properties are predicted by the formalism. (For a fuller and more pedagogical introduction to this point, see *e.g.* [15].) Similarly, the spectral action gives detailed guidance about the content and structure of the Higgs sector in models that go beyond the standard model of particle physics.

1.3 Extending to include nonassociativity: Why?

Over the past few centuries, we have become well accustomed to the fact that non-commutative structures are of central and ubiquitous importance in mathematics and physics. But, for most physicists, nonassociativity still carries a whiff of disreputability. We should start, then, by explaining the two motivations for studying nonassociativity in this paper: first a general motivation, followed by a more specific one.

Let us start with the more general motivation. The fundamental point is that, in the ordinary approach to physics, the basic input is a *symmetry group*: this is the starting point for specifying a model (like the standard model of particle physics). By contrast, in the spectral approach, the fundamental input is an *algebra*, and the symmetry group then emerges as the automorphism group of that algebra. Symmetry groups are associative by nature, but algebras are not. Just as some of the most beautiful and important groups are noncommutative, some of the most beautiful and important algebras (including Lie algebras, Jordan algebras and the Octonions) are nonassociative. Just as it would be unnatural to restrict our attention to commutative groups, it is unnatural to restrict our attention to associative algebras. In either case, imposing such an unnatural restriction amounts to blinding ourselves to something essential that the formalism is trying to tell us. From this standpoint, our task is to formulate the spectral approach to physics in such a way that the incorporation of nonassociative algebras becomes obvious and natural.

The more concrete motivation is the following. We would like to reformulate the most successful Grand Unified Theories (GUTs) – *e.g.* those based on $SU(5)$, $SO(10)$ and E_6 – in terms of the spectral action, but in order to do this, we are forced to use nonassociative input algebras. To appreciate this point, first note that the representation theory of associative \ast -algebras is much more restricted than the representation theory of Lie groups: Lie groups (like $SU(5)$) have an infinite number of irreps, but associative algebras (like the corresponding \ast -algebra $M_5(\mathbb{C})$ of 5×5 complex matrices, whose automorphism group is $SU(5)$) only have a finite number. In particular, if we ask whether key fermionic representations needed in GUT model building – such

as the **10** of $SU(5)$, the **16** of $SO(10)$, or the **27** of E_6 – are available as the irreps of algebras with the correct corresponding automorphism groups, the answer is “no” for associative algebras, and “yes” for nonassociative algebras. Furthermore, if we ask whether the exceptional groups (including E_6 which is of particular interest for GUT model building) appear as the automorphism groups of corresponding algebras, again the answer is “no” for associative algebras and “yes” for nonassociative algebras.

Finally, it is worth stressing that, even when the input \ast -algebra is nonassociative, at the end of the day one obtains an ordinary action describing ordinary physics: *i.e.* a gauge theory coupled to Einstein gravity, and built from ordinary (associative) scalar, spinor, gauge and metric fields, and living on ordinary spacetime. In other words, a reader who is unfamiliar with this topic might worry that the resulting theory is an exotic one which lives on some sort of nonassociative spacetime, or is built from nonassociative gauge field – but this is not what happens. In section 4 we present the first example of the spectral action evaluated on a nonassociative spectral triple, which serves to illustrate this point.

For earlier work on nonassociative geometry in somewhat different contexts, see [19–23].

1.4 Summary of key ideas and results

Here we summarize the contents of the paper, including the main ideas and results. Our first goal is to formulate the spectral action formalism in such a way that its meaning becomes clearer to physicists, and its extension to nonassociative \ast -algebras becomes obvious and natural. Our second goal is to present the simplest example of a nonassociative geometry, and evaluate the spectral action for this example.

We begin, in Section 2, by reviewing some of the mathematical background needed in the rest of the paper. In Subsection 2.1 we meet \ast -algebras, along with their automorphisms and derivations; and we meet the nonassociative \ast -algebra that will serve as our main example in this paper: \mathbb{O} , the algebra of Octonions. In Subsection 2.2, we review the basic axioms obeyed by a spectral triple in the ordinary associative case. We also give two examples of associative spectral triples satisfying these axioms, which will both be important later in the paper.

In Section 3, we discuss several of the main ideas needed to extend the spectral action formalism to incorporate nonassociative geometries:

- 1) First, in Subsection 3.1, we clarify what it means to represent a nonassociative \ast -algebra on a Hilbert space.
- 2) Second, in Subsection 3.2, we articulate the principle of \ast -automorphism covariance, which plays an important role in our approach. Usually, the automorphism invariance of the spectral action is presented as a consequence of the “spectral action

principle” (the principle that the action only depends on the spectrum of D which is, itself, invariant under automorphism). But in extending to the nonassociative case, we have found it important and clarifying to make a change of perspective, in which we elevate the $*$ -automorphism covariance to a fundamental underlying principle which then provides guidance about all subsequent steps. This principle ties together the transformations of the underlying algebra with those of the Hilbert space and all of the operators that act on it, in a way that is described in Subsection 3.2. The idea is that, in the spectral approach, the principle of $*$ -automorphism covariance subsumes and replaces the traditional covariance principles of physics: diffeomorphism covariance (in Einstein gravity) and gauge covariance (in gauge theory).

3) Third, in Subsection 3.3, we discuss the operators γ and J . J has two related roles that it retains in the nonassociative case: on the one hand, it defines a right action of the (representation of the) $*$ -algebra on the Hilbert space; and on the other hand, it extends the $*$ operation from the algebra to the Hilbert space itself. We explain how the usual “order zero” and “order one” conditions 2.8 in spectral geometry should be generalized in the nonassociative case. For example, in the nonassociative case, the order zero condition should continue to express the idea that, once we introduce right-action, the Hilbert space \mathcal{H} should be a bimodule over the $*$ -algebra \mathcal{A} (but now in the nonassociative sense [24]). Furthermore we argue that both J and γ should commute with the transformations $\hat{\alpha}$ induced by automorphisms α of the underlying $*$ -algebra \mathcal{A} . Stated another way: if δ represents a derivation of \mathcal{A} , and $\hat{\delta}$ is the operator that represents δ on \mathcal{H} , then we suggest that the conditions $[\hat{\delta}, \gamma] = 0$ and $[\hat{\delta}, J] = 0$ should be adopted as axiomatic in nonassociative geometry.

4) Fourth, in Subsection 3.4, we explain how to obtain the “fluctuated” Dirac operator D_* from the “unfluctuated” Dirac operator D . From our perspective, the “fluctuated” Dirac operator could also be called the “ $*$ -automorphism covariant” Dirac operator. The fundamental idea is that, just as in gauge theory one creates a covariant derivative by adding a one-form built from the generators of the underlying symmetry group, in spectral geometry, one creates a covariant Dirac operator by adding a ‘one-form’ built from the derivations of the underlying $*$ -algebra. This formulation reduces to the standard one in the usual case, and continues to make sense in the nonassociative case. In particular, this leads to a fluctuated Dirac operator that is still associative, even when the underlying $*$ -algebra is not. We give the explicit fluctuation when the non-associative algebra is alternative, as is the case for the octonion algebra: the fluctuations take the form of an element of the algebra of derivations, with arguments given by general hermitian one forms, and zero forms.

Finally, in Section 4, we present the simplest example of a nonassociative geometry, and explicitly evaluate the spectral action for this example. It is worth stressing that,

in this simple example, we can arrive at the spectral triple in two different ways. On the one hand, it satisfies all of the axioms for nonassociative geometry (including the nonassociative generalizations of the “order zero” and “order one” axioms) and is compatible with the principle of $*$ -automorphism covariance. On the other hand, it may alternatively be obtained by “twisting” an ordinary associative spectral triple (which satisfies the usual associative axioms). It is also worth stressing again that, although the input algebra is nonassociative, the resulting physical model is not: at the end of the day, the resulting action functional is an ordinary gauge theory, build from ordinary (associative) scalar, spinor, gauge and metric fields. In this paper, we just present the simplest possible model for illustration, and as a proof of principle. In a forthcoming paper [25], we show how to construct realistic models that include Higgs fields, spontaneous symmetry breaking and fermion masses.

2. Mathematical preliminaries

In this section, we review some of the mathematical background needed in the rest of the paper. In Subsection 2.1, we meet $*$ -algebras, along with their automorphisms and derivations; and we meet the the nonassociative $*$ -algebra that will serve as our main example in this paper: \mathbb{O} , the algebra of Octonions. In Subsection 2.2, we review the basic axioms obeyed by a spectral triple in the ordinary associative case. We also give two examples of associative spectral triples satisfying these axioms, which will both be important later in the paper.

2.1 $*$ -algebras, automorphisms, derivations, octonions

An algebra \mathcal{A} is a vector space over a field \mathbb{K} , which is equipped with an additional binary “product” operation: a \mathbb{K} -bilinear map from $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. (The product of two vectors $a, b \in \mathcal{A}$ is another vector $ab \in \mathcal{A}$.) A $*$ -algebra \mathcal{A} is an algebra that is equipped with an additional anti-linear involution map $*$: $\mathcal{A} \rightarrow \mathcal{A}$ satisfying

$$(a^*)^* = a, \quad (ab)^* = b^*a^*, \quad a, b \in \mathcal{A}. \quad (2.1)$$

In a $*$ -algebra \mathcal{A} : (i) the elements that satisfy $u^* = u^{-1}$ are called unitary; (ii) the elements that satisfy $h^* = h$ are called hermitian; and (iii) the elements that satisfy $a^* = -a$ are called anti-hermitian.

When we say an algebra is noncommutative, we mean that its product is noncommutative: $ab \neq ba$, $a, b \in \mathcal{A}$. Similarly, when we say an algebra is nonassociative, we mean that its product is nonassociative: $(ab)c \neq a(bc)$, $a, b, c \in \mathcal{A}$. Just as we introduce

the “commutator” $[a, b]$ to characterize the failure of commutativity, we introduce the “associator” $[a, b, c]$ to characterize the failure of associativity

$$[a, b] \equiv ab - ba, \quad [a, b, c] \equiv (ab)c - a(bc) \quad a, b, c \in \mathcal{A}. \quad (2.2)$$

Lie algebras are familiar examples of non-associative algebras. For example, consider the vector space of $N \times N$ complex anti-hermitian matrices. These do not form an algebra under ordinary matrix multiplication (since the ordinary matrix product of two anti-hermitian matrices is not, in general, anti-hermitian), but they do form an algebra if we define the product ab to be the commutator of the matrices a and b : $ab \equiv [a, b]$. The resulting algebra is a Lie algebra, since the product ab is anti-symmetric and satisfies the Jacobi identity $(ab)c + (bc)a + (ca)b = 0$; but it is easy to check that this product does not associate: $(ab)c \neq a(bc)$.

In order to describe algebras (especially nonassociative algebras) and their derivations, it is convenient to introduce the standard notation [24] in which L_a denotes the left-action of a , and R_a denotes the right-action of a :

$$L_a b \equiv ab \quad R_a b \equiv ba, \quad a, b \in \mathcal{A}. \quad (2.3)$$

In other words, L_a and R_a are two different linear operators on the vector space \mathcal{A} . As an illustration of this notation we can write $a((cv)b) = L_a R_b L_c v$ (with $a, b, c, v \in \mathcal{A}$). In particular, note that when \mathcal{A} is nonassociative, the left-hand side of this equation requires parentheses, but the right-hand side does not.

If \mathcal{A} is a $*$ -algebra, then an *automorphism* of \mathcal{A} is an invertible linear map $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ which respects the product and involution operations in \mathcal{A} :

$$\alpha(ab) = \alpha(a)\alpha(b) \quad \alpha(a^*) = (\alpha(a))^*, \quad (2.4)$$

and a *derivation* of \mathcal{A} is a linear map $\delta : \mathcal{A} \rightarrow \mathcal{A}$ which satisfies

$$\delta(ab) = \delta(a)b + a\delta(b) \quad \delta(a^*) = (\delta(a))^*. \quad (2.5)$$

Note that, when the automorphism α is infinitesimally close to the identity map “Id,” it can be written as $\alpha = \text{Id} + \delta$ where δ is a derivation. The derivations of \mathcal{A} are the infinitesimal generators of the automorphisms of \mathcal{A} ; they form a Lie algebra, with Lie bracket given by $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$ (where \circ denotes composition of functions).

For example, if $\mathcal{A} = C_\infty(\mathcal{M}, \mathbb{C})$ is the $*$ -algebra of smooth functions from a manifold \mathcal{M} to the complex numbers \mathbb{C} , then the automorphisms $\alpha_\varphi : \mathcal{A} \rightarrow \mathcal{A}$ are nothing but the maps $\alpha_\varphi(f) = f \circ \varphi$, where $f : \mathcal{M} \rightarrow \mathbb{C}$ is a smooth function and $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ is a diffeomorphism. And if we consider the diffeomorphisms infinitesimally close to

the identity, we see that the corresponding derivations have the form $\delta_v(f) = v^\mu \partial_\mu f$, where $v^\mu(x)$ is a contravariant vector field on \mathcal{M} .

Now consider an associative $*$ -algebra \mathcal{A} . Within the full group $\text{Aut}(\mathcal{A})$ of automorphisms of \mathcal{A} , there is a normal subgroup $\text{Inn}(\mathcal{A})$ of “inner automorphisms” of the form $\alpha_u(a) = uau^*$, where u is unitary. The group of “outer automorphisms” is then defined to be the quotient group $\text{Out}(\mathcal{A}) = \text{Aut}(\mathcal{A})/\text{Inn}(\mathcal{A})$. If we note that the unitary elements u are generated by anti-hermitian elements a ($u = e^a$) and study the inner automorphisms infinitesimally close to identity map, we find that the inner derivations (*i.e.* the generators of the inner automorphisms) have the form $\delta_a(b) = [a, b]$ or, equivalently,

$$\delta_a = L_a - R_a. \quad (2.6)$$

In the spectral reformulation of Einstein gravity coupled to the standard model of particle physics, the idea is (roughly) the following: one starts from a $*$ -algebra \mathcal{A} , whose full automorphism $\text{Aut}(\mathcal{A})$ corresponds to be the full (gauge+gravitational) symmetry group of the spectral action; roughly speaking, the inner automorphisms $\text{Inn}(\mathcal{A})$ correspond to the group of gauge transformations, while the outer automorphisms $\text{Out}(\mathcal{A})$ correspond to the group of gravitational symmetries (*i.e.* diffeomorphisms).

In this paper, for the purposes of illustration, we will focus on one of the most famous nonassociative algebras: namely, the algebra \mathbb{O} of octonions. The octonions occupy a special place in mathematics. They are one of only four normed division algebras: the real numbers \mathbb{R} , the complex numbers \mathbb{C} , the quaternions \mathbb{H} and the octonions \mathbb{O} . The algebras \mathbb{R} , \mathbb{C} , \mathbb{H} , and \mathbb{O} are respectively 1, 2, 4, and 8 dimensional with 0, 1, 3, and 7 imaginary elements which square to negative one. The octonions are the largest and most general algebra in the natural sequence $\mathbb{R} \in \mathbb{C} \in \mathbb{H} \in \mathbb{O}$, and they are intimately connected to some of the most beautiful structures in mathematics, including the exceptional Lie algebras and the exceptional Jordan algebra. For a nice expository introduction to the octonions, and their connections to other areas of mathematics, see [26]. Here, let us note three features in particular. (i) First, the octonions are an example of an “alternative algebra” – *i.e.* an algebra in which the associator $[a, b, c]$ flips sign under interchange of any two of its arguments. (ii) Second, for any alternative algebra \mathcal{A} , including the octonions, the general derivation $\delta : \mathcal{A} \rightarrow \mathcal{A}$ may be written as a linear combination of derivations of the form [27]

$$\delta_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b] \quad a, b \in \mathcal{A}. \quad (2.7)$$

(iii) Third, the algebra of derivations of \mathbb{O} is \mathfrak{g}_2 (the smallest exceptional Lie algebra) and the automorphism group of \mathbb{O} is G_2 (the smallest exceptional Lie group).

2.2 Associative spectral triples (including two important examples)

In Subsection 1.1, we met the idea that a spectral triple is a collection of objects $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$ which, together, specify a spectral geometry. In this Subsection, we review several of the key aspects of spectral triples that will be important later in the paper: (i) first, we will describe the key axioms relating various elements of the spectral triple $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$; (ii) second, we introduce the two particular examples of spectral triples that will be important to us in this paper; and (iii) finally, we will describe how to take the product of two spectral triples to obtain a third.

In order to form a spectral triple, the five elements $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$ must not be chosen arbitrarily. Instead, according to the axioms of spectral geometry, they must satisfy certain conditions that give the spectral triple its structure. Here we merely summarize these conditions; for an explanation of their meaning, see [2, 3, 16]; for pedagogical introductions aimed at physicists, see [14, 15].

To describe the structure of a spectral triple, it is helpful to think of building it up in five steps, adding one element at a time. (i) In Step 1, we choose a $*$ -algebra \mathcal{A} . (ii) In Step 2, we represent \mathcal{A} on the Hilbert space \mathcal{H} : the representation is a map π which takes each element $a \in \mathcal{A}$ to a corresponding operator $\tilde{a} \equiv \pi(a)$ that acts on \mathcal{H} . The map π must preserve the structure of \mathcal{A} : *i.e.* it must be linear and satisfy $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a^*) = (\pi(a))^*$. In the next three steps, we add three more operators that act on \mathcal{H} (namely, D , γ and J) and describe the constraints that these operators must satisfy to form a spectral triple. (iii) In Step 3, we add the hermitian operator D : the commutator $[D, \pi(a)]$ must be bounded ($\forall a \in \mathcal{A}$). (iv) In Step 4, we add the hermitian and unitary operator γ : it must satisfy $\{\gamma, D\} = 0$ and $[\gamma, \pi(a)] = 0$ ($\forall a \in \mathcal{A}$). (v) In Step 5, we add the anti-unitary operator J : it satisfies $J^2 = \epsilon$, $JD = \epsilon' DJ$, and $J\gamma = \epsilon'' \gamma J$, where ϵ , ϵ' and ϵ'' denote three distinct \pm signs determined by the so-called “KO-dimension” of the spectral triple¹. Furthermore, given any operator $\tilde{a} = \pi(a)$, we can use J to define a dual operator $\tilde{a}^0 = J\tilde{a}^*J^*$; the interpretation is that, for any $a \in \mathcal{A}$, it is represented in two ways: as an operator \tilde{a} that acts on \mathcal{H} from the left, and an operator \tilde{a}^0 that acts on \mathcal{H} from the right. In the normal case, where \mathcal{A} is an *associative* $*$ -algebra, one then imposes two additional requirements:

$$[\tilde{a}, \tilde{c}^0] = 0 \quad \forall \{a, c\} \in \mathcal{A} \quad \text{“the order zero condition,”} \quad (2.8a)$$

$$[[D, \tilde{a}], \tilde{c}^0] = 0 \quad \forall \{a, c\} \in \mathcal{A} \quad \text{“the order one condition.”} \quad (2.8b)$$

As we will see below, these last two axioms (the order-zero and order-one conditions) naturally become modified when the underlying algebra \mathcal{A} is nonassociative.

¹See Definition 16 in section 6.8 of [28], or section 2.2.2 in [15].

Let us consider two examples of spectral triples satisfying the above axioms. Both of these examples will play an important role as background for the nonassociative example treated later in the paper.

Example 1. The first example is the canonical spectral triple M introduced in section 1.1.

Example 2. To explain the second example, we must first present a few more pieces of mathematical background. The field \mathbb{K} and the group G together define a natural algebra: namely, the “group algebra” $\mathbb{K}G$ consisting of arbitrary linear combinations of the form $\sum_i k_i g_i$, where $k_i \in \mathbb{K}$ and $g_i \in G$. These elements may be added and multiplied in the obvious way, and thus form an algebra over the field \mathbb{K} ; the dimension of the algebra $\mathbb{K}G$ is just the order of the group G . $\mathbb{K}G$ is naturally a $*$ -algebra, with the $*$ operation given by $(\sum_i k_i g_i)^* = \sum_i k_i^* g_i^{-1}$; and it is also naturally a Hilbert space, with the inner product of two vectors $v^{(1)} = \sum_i k_i^{(1)} g_i$ and $v^{(2)} = \sum_i k_i^{(2)} g_i$ given by $\langle v^{(1)} | v^{(2)} \rangle = \sum_i (k_i^{(1)*}) k_i^{(2)}$. Thus, it is natural to consider a spectral triple $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$ where \mathcal{A} and \mathcal{H} are both given by $\mathbb{K}G$, and \mathcal{A} is represented in the obvious way: *i.e.* π is the identity map (so $\tilde{a} = a$), and the action of the operator \tilde{a} on an element of \mathcal{H} is given by the ordinary product in $\mathbb{K}G$. Furthermore, we can take $\gamma = 1$; the condition $\{\gamma, D\} = 0$ then implies $D = 0$. Finally, the action of J on \mathcal{H} is naturally given by the $*$ -operation in $\mathbb{K}G$: $Jv = J(\sum_i k_i g_i) = (\sum_i k_i g_i)^* = \sum_i k_i^* g_i^{-1}$. More specifically, in this paper we will be particularly interested in the case where $\mathbb{K} = \mathbb{R}$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, so that $\mathbb{K}G$ is an 8-dimensional algebra over the real numbers. This algebra is of special interest because it is related by a twist to the flagship non-associative algebra: \mathbb{O} , the algebra of octonions (see section 3.3).

One can check that both of these example spectral triples satisfy the list of conditions outlined above.

It is important to note that, given two spectral triples, $T_1 = \{\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1, J_1\}$ and $T_2 = \{\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2\}$, we can construct a third spectral triple, $T_{12} = T_1 \times T_2$, which is a product of the previous two, as follows: $T_{12} = \{\mathcal{A}_{12}, \mathcal{H}_{12}, D_{12}, \gamma_{12}, J_{12}\}$, where $\mathcal{A}_{12} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $D_{12} = D_1 \otimes \mathbb{I}_2 + \gamma_1 \otimes D_2$, $\gamma_{12} = \gamma_1 \otimes \gamma_2$ and $J_{12} = J_1 \otimes J_2$.

In particular, one is often interested in the product $M \times F$ of the infinite-dimensional canonical spectral triple M with a finite-dimensional spectral triple F . When the finite-dimensional algebra \mathcal{A}_F in F is noncommutative, $M \times F$ is referred to as an “almost commutative geometry.” Similarly, when the finite-dimensional algebra \mathcal{A}_F in F is nonassociative, we will refer to $M \times F$ as an “almost-associative geometry.” The spectral reformulation of Einstein gravity coupled to the standard model of particle physics is based on an almost commutative geometry; and in Section 4 we will compute the spectral action for an almost-associative geometry.

3. Nonassociative geometry

3.1 Representing a nonassociative \ast -algebra

The starting point for the spectral formalism is a \ast -algebra \mathcal{A} , that is “represented” on a Hilbert space \mathcal{H} . Let us be careful to explain what this means when \mathcal{A} is nonassociative.

A Hilbert space \mathcal{H} is a vector space over \mathbb{K} , that is equipped with an inner product $\langle \cdot | \cdot \rangle$ – a rule for multiplying two vectors a and c to get a scalar $\langle a | c \rangle \in \mathbb{K}$. The inner product is skew-linear in its first argument, linear in its second argument, skew-symmetric ($\langle a | c \rangle = \langle c | a \rangle^*$), and positive definite ($\langle a | a \rangle \geq 0$). A “representation” of a \ast -algebra \mathcal{A} on a Hilbert space \mathcal{H} is a map π which takes each element $a \in \mathcal{A}$ to a corresponding linear operator $\pi(a) = \tilde{a}$. The linear operator \tilde{a} maps each vector $v \in \mathcal{H}$ to a new vector $\tilde{a}v \in \mathcal{H}$. The map π must preserve the structure of \mathcal{A} : in particular, it should be linear, and satisfy $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a^*) = (\pi(a))^*$.

In the case where \mathcal{A} is associative, the product $\tilde{a}\tilde{b}$ just denotes the composition of the operators \tilde{a} and \tilde{b} ; this is an associative product, so expressions like $\tilde{a}\tilde{b}\tilde{c}$ and $\tilde{a}\tilde{b}v$ are unambiguous, and do not require any additional parentheses. By contrast, in the case where \mathcal{A} is nonassociative, the operator \tilde{a} has two different roles that should be carefully distinguished: on the one hand it can operate on a vector $v \in \mathcal{H}$, mapping it to a new vector $\tilde{a}v \in \mathcal{H}$; on the other hand, it can multiply another operator \tilde{b} to form a third operator $(\tilde{a}\tilde{b})$. It is important to note that, since the operators \tilde{a} and \tilde{b} represent elements in an underlying nonassociative algebra \mathcal{A} , their product $(\tilde{a}\tilde{b})$ will *not* be given by the composition of the operators \tilde{a} and \tilde{b} (which is associative); instead, it will be given by some other product that reflects the nonassociativity of \mathcal{A} . In particular, we must be careful to remember that $(\tilde{a}\tilde{b})\tilde{c} \neq \tilde{a}(\tilde{b}\tilde{c})$ and $\tilde{a}(\tilde{b}v) \neq (\tilde{a}\tilde{b})v$.

As a simple illustration, consider the case where \mathcal{A} is a nonassociative \ast -algebra equipped with a natural inner product $\langle \cdot | \cdot \rangle$ (so that it may also be interpreted as a Hilbert space \mathcal{H}). Then \mathcal{A} may be “represented on itself” in an obvious way: we take the Hilbert space \mathcal{H} to be the same as the \ast -algebra \mathcal{A} ; we take the algebra homomorphism π to be the identity map ($\tilde{a} = a$); and we take the product of two operators \tilde{a} and \tilde{b} , or the action of an operator \tilde{a} on a Hilbert space element v , to be given by the underlying product in \mathcal{A} : $\tilde{a}\tilde{b} = ab$, $\tilde{a}v = av$. For concreteness, we can imagine that $\mathcal{A} = \mathbb{O}$, which is equipped with a natural inner product $\langle a | b \rangle = (1/2)(a^*b + b^*a) = \text{Re}(a^*b)$ where a^* denotes the octonionic conjugate of a .

3.2 The principle of automorphism covariance

Consider an automorphism α of the \ast -algebra \mathcal{A} , which maps each element $a \in \mathcal{A}$ to a new element $a' \in \mathcal{A}$. This induces a corresponding transformation $\tilde{\alpha}$ that maps each

operator \tilde{a} to a new operator \tilde{a}' , and a corresponding transformation $\hat{\alpha}$ that maps each vector $v \in \mathcal{H}$ to a new vector $v' \in \mathcal{H}$:

$$a \rightarrow a' = \alpha(a) \quad (3.1a)$$

$$\tilde{a} \rightarrow \tilde{a}' = \tilde{\alpha}(\tilde{a}) \quad (3.1b)$$

$$v \rightarrow v' = \hat{\alpha}(v) \quad (3.1c)$$

To tie the transformations α , $\tilde{\alpha}$ and $\hat{\alpha}$ together, we demand that they satisfy the *principle of automorphism covariance*, which demands that our whole formalism should “commute” with automorphisms of the underlying \ast -algebra. In other words, any sensible expression should have the property that, if we first transform its components and then evaluate the expression, this should be the same as first evaluating the expression and then transforming the result.

For starters, we apply the principle to the expression $\tilde{a} = \pi(a)$: it requires that $\pi(\alpha(a)) = \tilde{\alpha}(\pi(a))$, $\forall a \in \mathcal{A}$; or, in other words:

$$\pi \circ \alpha = \tilde{\alpha} \circ \pi \quad (3.2)$$

where \circ denotes composition of functions. Next, we apply the principle to the expression $\tilde{a}v$: it requires that $\hat{\alpha}(\tilde{a}v) = \tilde{\alpha}(\tilde{a})\hat{\alpha}(v)$; or, in other words:

$$\tilde{a}' = \tilde{\alpha}(\tilde{a}) = \hat{\alpha} \circ \tilde{a} \circ \hat{\alpha}^{-1} \quad \forall a \in \mathcal{A}. \quad (3.3)$$

For illustration, consider the simple example given at the end of Subsection 3.1. In this case, we would have $\alpha = \tilde{\alpha} = \hat{\alpha}$, and all of the above equations would be automatically satisfied.

As we have seen, there are three other important operators which act on \mathcal{H} : namely, D , γ , and J . Applying the principle to the expressions Dv , γv and Jv we see that, under an automorphism α , these operators must transform as

$$D \rightarrow D' = \hat{\alpha} \circ D \circ \hat{\alpha}^{-1} \quad (3.4a)$$

$$\gamma \rightarrow \gamma' = \hat{\alpha} \circ \gamma \circ \hat{\alpha}^{-1} \quad (3.4b)$$

$$J \rightarrow J' = \hat{\alpha} \circ J \circ \hat{\alpha}^{-1} \quad (3.4c)$$

In fact, as we shall see, J and γ are naturally invariant under this transformation (*i.e.* $J' = J$ and $\gamma' = \gamma$). But D is not invariant: instead, the automorphisms of the underlying \ast -algebra \mathcal{A} induce a transformation or “fluctuation” of D .

We take the principle of automorphism covariance to be a fundamental principle lying at the base of the spectral reformulation of physics: as we shall see, it replaces

(or subsumes or implies) the more familiar principles of covariance under coordinate transformations and gauge transformations, which are usually taken as the starting points for Einstein gravity and gauge theory. We will also see that this principle will give us all the guidance we need in formulating the spectral action principle unambiguously, even when \mathcal{A} is nonassociative.

3.3 The Real Structure J , and the \mathbb{Z}_2 grading γ

A spectral triple is said to be “real” if it is equipped with a real structure operator J and “even” if it is equipped with a \mathbb{Z}_2 grading operator γ . In this section we will discuss the generalization of both operators to the non-associative setting, beginning with the operator J . For a more complete exposition in the associative case see references [2, 3].

As mentioned in section 1.1 one can think of J as extending the $*$ operation from the algebra to the Hilbert space. This extension provides a right action $\pi_a^0 = J\tilde{a}^*J^*$, $a \in \mathcal{A}$, and in the associative case \mathcal{H} is extended from a left module over \mathcal{A} to a bi-module over \mathcal{A} . Further, Connes introduces the order zero and order one conditions, which we have provided in equations (2.8a), and (2.8b) respectively. These conditions define the associative bi-module structure of the Hilbert space \mathcal{H} , and ensure that the Dirac operator D acts as a first order differential operator.

The Order Zero Condition (2.8a) presents an apparent hurdle in extending the framework of NCG to non-associative algebras. This may not be immediately clear so we will make use of an example to elucidate the point. Consider the case where as input data we take the algebra of octonions acting naturally on themselves $\mathcal{A} = \mathcal{H} = \mathbb{O}$. In this case we find

$$[L_b^0, L_{\tilde{a}}]\tilde{v} = [JL_{\tilde{b}}^*J^*, L_{\tilde{a}}]\tilde{v} = [\tilde{a}, \tilde{v}, \tilde{b}] \neq 0, \quad a, b \in \mathcal{A}, \quad v \in \mathcal{H} \quad (3.5)$$

where J is octonionic conjugation and $L_b^0 := JL_{\tilde{b}}^*J^*$. For a non-associative algebra the associator is typically non-zero. We see that the Order Zero Condition as given in equation (2.8a) is incompatible with the representation of the octonions on themselves, which is arguably their most natural form. This same ‘problem’ will exist more generally for all non-associative algebras and their representations. Fortunately what is known as the quasialgebra structure of the octonions hints at a possible way to get around this issue [23, 29–31].

A knowledge of quasialgebras will not be necessary to follow the remaining discussion, although we recommend them as very interesting! For our present purposes a quasialgebra can be thought of as an algebra that is, in some well defined way, related to certain other quasialgebra structures. Specifically, starting with an associative quasialgebra (\mathcal{A}, \cdot) , we can perform what is known as a ‘twist’ to obtain a new quasialgebra (\mathcal{A}_F, \times) . The new algebra \mathcal{A}_F shares the same underlying vector space as \mathcal{A} but

has a new product \times . It is possible in this way to describe the non-associativity of a quasialgebra (\mathcal{A}_F, \times) as resulting from a ‘twist’ from an associative quasialgebra (\mathcal{A}, \cdot) .

As an example, consider again our input algebra $\mathcal{A}_F = \mathcal{H}_F = \mathbb{O}$. The authors Albuquerque and Majid [29] have already described in full detail the octonions as a quasialgebra resulting from a ‘twist’ on the associative group algebra $\mathbb{K}G$, where $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (an algebra we discussed in section 2.2). Note that here we have added the subscript ‘ F ’ to indicate that our algebra is being considered as a ‘twisted’ structure. The notation is a convenient coincidence as this twisted algebra will also serve as our finite algebra later on.

We can write each basis element of $\mathbb{K}G$ in the form $g_i = (i_1, i_2, i_3)$, where $i_j \in \{0, 1\}$; and then $\mathbb{K}G$ simply inherits the group multiplication law: $j \cdot k$ simply means adding the two vectors $(j \text{ and } k)$, mod 2. From here, we can obtain the octonions by performing a ‘twist’ – i.e. by replacing the multiplication law $x \cdot y$ with the new multiplication law:

$$g_i \times g_j = g_i \cdot g_j F(g_i, g_j), \quad \forall g_i, g_j \in G \quad (3.6)$$

where F is known as a ‘2-cochain twist’ taking values in the field \mathbb{K} over which the algebra \mathcal{A}_F is defined. The 2-cochain F is given in our case as [29]

$$F(g_i, g_j) = (-1)^f, \quad f = i_1(j_1 + j_2 + j_3) + i_2(j_2 + j_3) + i_3 j_3 + j_1 i_2 i_3 + i_1 j_2 i_3 + i_1 i_2 j_3. \quad (3.7)$$

We are now in a position to analyse how the order zero condition behaves under a ‘twist’ from the associative $\mathcal{A} = \mathcal{H} = \mathbb{K}G$ to the nonassociative $\mathcal{A}_F = \mathcal{H}_F = \mathbb{O}$. As \mathcal{A} is associative it will satisfy the order zero condition given in (2.8a).

$$\begin{aligned} [\pi_{g_j}^0, \pi_{g_i}] \tilde{g}_k &= (\tilde{g}_i \cdot \tilde{g}_k) \cdot \tilde{g}_j - \tilde{g}_i \cdot (\tilde{g}_k \cdot \tilde{g}_j) = 0, \quad g_i, g_j, g_k \in G, \\ \text{‘twist’} \rightarrow 0 &= F^{-1}(g_i, g_k) F^{-1}(g_i \cdot g_k, g_j) (\tilde{g}_i \times \tilde{g}_k) \times \tilde{g}_j \\ &\quad - F^{-1}(g_i, g_k \cdot g_j) F^{-1}(g_k, g_j) \tilde{g}_i \times (\tilde{g}_k \times \tilde{g}_j) \\ &= \frac{F(g_i, g_k \cdot g_j) F(g_k, g_j)}{F(g_i, g_k) F(g_i \cdot g_k, g_j)} (\tilde{g}_i \times \tilde{g}_k) \times \tilde{g}_j - \tilde{g}_i \times (\tilde{g}_k \times \tilde{g}_j) \\ &= \Phi_{\tilde{g}_i, \tilde{g}_k, \tilde{g}_j}^{-1} (\tilde{g}_i \times \tilde{g}_k) \times \tilde{g}_j - \tilde{g}_i \times (\tilde{g}_k \times \tilde{g}_j) \\ &:= [L_{\tilde{g}_j}^0, L_{\tilde{g}_i}]_{\Phi} \tilde{g}_k \end{aligned} \quad (3.8)$$

where the ‘associator’ is defined as $\Phi_{\tilde{g}_i, \tilde{g}_k, \tilde{g}_j} := \frac{F(g_i, g_k) F(g_i \cdot g_k, g_j)}{F(g_i, g_k \cdot g_j) F(g_k, g_j)}$. After the twist we should consider the basis elements $g_i, g_j \in \mathcal{A}_F$ and $g_k \in \mathcal{H}_F$. Equation (3.8) suggests we introduce an augmented order zero condition in the general sense given by

$$[L_b^0, L_{\tilde{a}}]_{\Phi} = 0 \quad \forall a, b \in \mathcal{A}_F. \quad (3.9)$$

Here the subscript Φ can be seen as telling us to ‘flip’ the brackets on one side of the commutator when acting on a hilbert space element. Note that for an associative algebra, the ‘associator’ Φ will be trivial and our augmented conditions will collapse back to those given in the associative case (2.8a), (2.8b). Although we will not need to make much use of it in this paper, using a similar argument we could generalize the order one condition.

$$[[D, L_{\bar{b}}], L_a^0]_{\Phi} = 0 \quad \forall a, b \in \mathcal{A}_F \quad (3.10)$$

The augmented order one condition will become important when we begin to discuss more complicated models in a follow-up paper, and we will discuss it more thoroughly there. For now simply note that if D is a first order derivation on the algebra \mathcal{A} and \mathcal{H} as given in [24] then $[D, L_a] = L_{Da}$. In this case the augmented order one condition is expected to hold following similar steps as in equation (3.8).

What is the meaning of the augmented order zero and order one conditions given in equations (3.9), and (3.10)? The precise meaning is exactly as it was in the associative case. That is, the operator J defines the bi-module structure of the Hilbert space \mathcal{H} . In the non-associative case however we need to be careful to explain exactly what we mean by a bi-module.

Given a class of (nonassociative) algebras \mathcal{C} satisfying a set of multilinear identities $I_i(a_1, \dots, a_{n_i}) = 0$, then \mathcal{H} is a bimodule over \mathcal{A} in \mathcal{C} if all of the identities, obtained by replacing any single $a_j \in \mathcal{A}$ by any $v \in \mathcal{H}$, are satisfied [24]. In other words, the order zero condition simply states that the bimodule \mathcal{H} must have the same ‘(non)associative’ properties as the objects over which it acts. So, in the associative case, the order zero condition is nothing more than the statement that \mathcal{H} ‘associates’. Similarly, our augmented order zero condition is nothing more than the statement that \mathcal{H} is governed by the same associator as \mathcal{A} . The augmented order one condition again ensures that the operator D is first order.

In discussing the twist from $\mathcal{A} = \mathbb{K}G$ to $\mathcal{A}_F = \mathbb{O}$ the authors Albuquerque and Majid [29] give a ‘natural involution’ ($*$ operation) on the twisted algebra basis

$$Je_i = F(e_i, e_i)e_i \quad (3.11)$$

From equation (3.7) it can be seen that this involution is simply octonionic conjugation. Prior to twisting we can simply take $F(e_j, e_i) = 1, \forall e_i \in \mathbb{K}G$. Notice that in $\mathbb{K}G$ each basis element is its own inverse. For this reason the ‘natural’ $*$ operation coincides in the untwisted case with what is known as the ‘antipode’ operator S on $\mathbb{K}G$:

$$Je_i = Se_i = e_i^{-1} \quad (3.12)$$

We can consider the data $\{\mathcal{A}, \mathcal{H}, J\} = \{\mathbb{O}, \mathbb{O}, J_F\}$ as being ‘twisted’ from the data $\{\mathbb{K}G, \mathbb{K}G, S\}$.

Now let us turn to the \mathbb{Z}_2 grading γ . This linear operator is both hermitian ($\gamma^* = \gamma$) and unitary ($\gamma^* = \gamma^{-1}$). Hence it satisfies $\gamma^2 = 1$, so its eigenvalues are ± 1 , and it correspondingly decomposes \mathcal{H} into two subspaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. For physicists, the familiar example is Dirac’s helicity operator γ_5 which has the above properties and decomposes the space of Dirac spinors into positive and negative (helicity) subspaces: $L^2(\mathcal{M}, S) = L^2_+(\mathcal{M}, S) + L^2_-(\mathcal{M}, S)$.

One nice way to think of γ_5 is as a volume form. It is this perspective that is most readily generalisable to the non-commutative and non-associative cases. Recall that on a spin manifold the Dirac operator is given by $\not{D} = -i\gamma^\mu \nabla_\mu^S$, where the γ^μ are the Dirac Gamma matrices, and ∇_μ^S is the spin connection. Although this Dirac operator may be unbounded, its commutator with elements of the algebra of functions over the manifold $df = [\not{D}, f] = -i\gamma^\mu (\partial_\mu f)$ is bounded. In fact this bounded operator gives the Clifford representation of the 1-form $df = dx^\mu (\partial_\mu f)$ [15]. Similarly, we see that the γ_5 grading operator in the canonical case can be considered as the Clifford representation of a volume form.

$$\frac{1}{4!} \epsilon_{\mu\nu\tau\rho} \gamma^\mu \gamma^\nu \gamma^\tau \gamma^\rho = \gamma^1 \gamma^2 \gamma^3 \gamma^4 := \gamma_5. \quad (3.13)$$

Connes generalized the grading structure to non-commutative even dimensional orientable spin manifolds [3]. In fact, Connes introduced an entirely new differential calculus in which he generalized the De Rham cohomology of ordinary differential calculus to what is known as cyclic cohomology [16, 32]. When extending to the non-associative case further generalization needs to be made. Fortunately much work has already been completed in developing an appropriate differential calculus. As a description of this generalization will not be necessary for understanding our first example non-associative geometry we will not give an account of it here, and instead refer to the interested reader to the literature [23, 33].

Both the real structure J and the \mathbb{Z}_2 grading γ should be compatible with the automorphisms of the underlying $*$ -algebra: automorphisms should not effect the split between positive and negative helicity states, or between particles and anti-particles. We can express this requirement in terms of automorphisms:

$$\gamma' = \hat{\alpha} \circ \gamma \circ \hat{\alpha}^{-1} = \gamma, \quad (3.14a)$$

$$J' = \hat{\alpha} \circ J \circ \hat{\alpha}^{-1} = J. \quad (3.14b)$$

or in terms of the derivations that generate them

$$[\hat{\delta}, \gamma] = 0 \quad (3.15a)$$

$$[\hat{\delta}, J] = 0 \quad (3.15b)$$

Readers can convince themselves that these conditions hold in the associative case, and in the nonassociative example discussed below. We propose that it is natural to take these conditions to be true more generally; *i.e.* to take them as axiomatic in nonassociative geometry.

3.4 Fluctuating the Dirac Operator D

In ordinary gauge theory, the principle of gauge covariance leads us to replace the partial derivative ∂_μ by the gauge covariant derivative $D_\mu = \partial_\mu + A_\mu$, which is ultimately the object from which we build a gauge-invariant action. In a closely analogous way, in spectral geometry, the principle of $*$ -automorphism covariance leads us to replace the fiducial “Dirac operator” D with the “fluctuated” or “ $*$ -algebra covariant” Dirac operator D_* , which is ultimately the object from which we build the $*$ -automorphism-invariant spectral action.

It is helpful, then, to warm up by reviewing the story in ordinary gauge theory. We can write a general gauge transformation in the form $u(x) = \exp[\alpha^a(x)T_a]$, where T_a are the generators of the gauge group. Now consider a multiplet of matter fields ψ that transforms covariantly under a gauge transformation: $\psi \rightarrow \psi' = u\psi$. We would like to introduce a gauge-covariant derivative operator D_μ with the property that $D_\mu\psi$ also transforms covariantly: $D_\mu\psi \rightarrow D'_\mu\psi' = uD_\mu\psi$. In other words, we want D_μ to transform as

$$D_\mu \rightarrow D'_\mu = uD_\mu u^{-1}. \quad (3.16)$$

Start with the special case where $D_\mu = \partial_\mu$, and perform an infinitesimal gauge transformation to obtain $D'_\mu = \partial_\mu - [\partial_\mu, \alpha^a(x)]T_a$. By inspection of this formula, we can see that in the general case we can take

$$D_\mu = \partial_\mu + A_\mu \quad \text{where} \quad A_\mu = A_\mu^a T_a, \quad (3.17)$$

where A_μ^a are arbitrary gauge fields (one for each linearly independent generator T_a). To make D_μ transform as in Eq. (3.16), we should take A_μ to transform as

$$A_\mu \rightarrow A'_\mu = uA_\mu u^{-1} + u[\partial_\mu, u^{-1}]. \quad (3.18)$$

Now let us present the analogous story in associative spectral geometry. Consider an element $v \in \mathcal{H}$; under an inner $*$ -automorphism of \mathcal{A} , it transforms as $v \rightarrow v' = \hat{\alpha}(v)$ (For the relationship between the hatted and unhatted transformations, see Subsection 3.2). We would like to introduce a $*$ -automorphism-covariant Dirac operator D_* such that D_*v also transforms as $D_*v \rightarrow D'_*v' = \hat{\alpha}(D_*v)$. In other words, we want D_* to transform as

$$D_* \rightarrow D'_* = \hat{\alpha} \circ D_* \circ \hat{\alpha}^{-1}. \quad (3.19)$$

As discussed in section 2.1 the inner automorphisms for an associative \ast -algebra \mathcal{A} are generated by elements of the algebra of derivations $\tilde{\delta}_c = L_{\tilde{c}} - R_{\tilde{c}}$. We therefore have

$$D'_\ast = e^{\tilde{\delta}_c} D_\ast e^{-\tilde{\delta}_c} \quad (3.20)$$

which to first order can be written as

$$\begin{aligned} D'_\ast &\simeq D_\ast - [D_\ast, \tilde{\delta}_c] \\ &= D_\ast - [D_\ast, c^k \tilde{e}_k] - \epsilon' J[D_\ast, c^k \tilde{e}_k] J^\ast, \end{aligned} \quad (3.21)$$

where the factor ϵ' , is dependent on the real structure of the spectral triple: $D_\ast J = \epsilon' J D_\ast$. In the final line we have expressed the derivation elements in terms of the basis elements of the algebra representation $\tilde{c} = c^k \tilde{e}_k \in \pi(\mathcal{A})$. As an aside, for the special case of an Almost-Commutative geometry, the coefficients c^k will be spatially dependent and equation (3.21) can be split in a rather suggestive way.

$$D'_\ast \simeq D_\ast - \underbrace{[D_\ast, c^k] \tilde{e}_k - \epsilon' J[D_\ast, c^k] \tilde{e}_k J^\ast}_{\text{Gauge inhomogeneous terms}} - \underbrace{c^k [D_\ast, \tilde{e}_k] - \epsilon' J c^k [D_\ast, \tilde{e}_k] J^\ast}_{\text{Higgs inhomogeneous terms}} \quad (3.22)$$

We will continue the discussion of the split in equation (3.22) in a later paper [25] when we explore models with non-trivial Higgs fields. Returning to the present discussion, the term $[D, c]$ in equation (3.21) is a one form [15]. By inspection, the fluctuated dirac operator should be of the form

$$D_\ast = D + A_{(1)} + \epsilon' J A_{(1)} J^\ast := D + B \quad (3.23)$$

where D is the un-fluctuated Dirac operator. The generalized hermitian one form in equation (3.23) is given by Connes as $A_{(1)} = \sum \tilde{a}[D, \tilde{b}]$, where the sum is given over elements $a, b \in \mathcal{A}$. The ‘fluctuation’ term B is analogous to the connection term that appears in equation (3.17). As it will be of importance later, notice that the fluctuation is given as the adjoint action of the one form $A_{(1)}$. Notice also that if D acts as a derivation on the algebra representation and the Hilbert space, then the generalized one form A can be written as

$$A_{(1)} = \sum L_{\tilde{a}(D\tilde{b})}, \quad (3.24)$$

Under an algebra automorphism the fluctuation term transform as

$$B \rightarrow \hat{\alpha} B \hat{\alpha}^{-1} + \hat{\alpha} [D, \hat{\alpha}^{-1}]. \quad (3.25)$$

Thus the fluctuation B ensures that the fluctuated Dirac operator D_\ast transforms in the desired way under inner automorphisms, as given in equation (3.19). We would

now like to discuss fluctuating the Dirac operator in the case where the input algebra is no-longer associative. Again the fluctuated Dirac operator should transform under inner $*$ -automorphisms of the input algebra as shown in equation (3.19). The only difference is that now the automorphisms will be generated by elements of the algebra of derivations $D(\pi(\mathcal{A}))$ for a non-associative algebra.

In this paper we will eventually give an example based on the octonions. The octonions are an alternative algebra, and so their $*$ -automorphisms will be generated by derivation elements for an alternative algebra. A general derivation will be given by an arbitrary sum of elements $\tilde{\delta}_{b,c} = [L_{\tilde{b}}, L_{\tilde{c}}] + [L_{\tilde{b}}, R_{\tilde{c}}] + [R_{\tilde{b}}, R_{\tilde{c}}] \in D(\pi(\mathcal{A}))$ (see equation (2.7)). To first order the fluctuated Dirac operator must transform as

$$\begin{aligned} D'_* &\simeq D_* - [D_*, \tilde{\delta}_{b,c}] \\ &= D_* - [[D_*, L_{\tilde{b}}], L_{\tilde{c}}] + [[D_*, L_{\tilde{b}}], JL_{\tilde{c}}J^*] - \epsilon' J[[D_*, L_{\tilde{b}}], L_{\tilde{c}}]J^* \\ &\quad + [[D_*, L_{\tilde{c}}], L_{\tilde{b}}] - \epsilon' [J[D_*, L_{\tilde{c}}]J^*, L_{\tilde{b}}] + \epsilon' J[[D_*, L_{\tilde{c}}], L_{\tilde{b}}]J^*, \end{aligned} \quad (3.26)$$

where comparison between equations (3.21) and (3.26) should be stressed. Once again the fluctuated Dirac operator will be given by

$$D_* = D + B, \quad (3.27)$$

where D is the un-fluctuated Dirac operator, and the fluctuation term B once again transforms as

$$B \rightarrow \hat{\alpha} B \hat{\alpha}^{-1} + \hat{\alpha} [D, \hat{\alpha}^{-1}]. \quad (3.28)$$

Just as is done in the associative case, the general form for the fluctuation term in equation (3.27) comes from inspection of the inhomogeneous terms in equation (3.26):

$$B = \sum \delta_{A_{(1)}, A_{(0)}} := \sum [A_{(1)}, A_{(0)}] - [A_{(1)}, JA_{(0)}J^*] + \epsilon' J[A_{(1)}, A_{(0)}]J^*, \quad (3.29)$$

where the sum is taken over generalized hermitian ‘one forms’ $A_{(1)}$, and generalized ‘zero forms’ $A_{(0)}$. The ‘zero forms’ $A_{(0)}$ will simply be given by left acting elements of the alternative algebra. The generalized ‘one forms’ will depend on the representation of the algebra π , the real structure J , and the form of the un-fluctuated Dirac operator D . In the important case where D acts as a derivation on the input algebra and Hilbert space however, we have in comparison with equation (3.24)

$$A_{(1)} = \sum L_{\tilde{a}(D\tilde{b})}, \quad a, b \in \mathcal{A}. \quad (3.30)$$

The fluctuations given in equation (3.29) are a natural generalization of the fluctuations given in equation (3.23) for the associative case. Because all associative algebras are

alternative, the reader can indeed check for themselves that (3.29) reduces to the associative case when the input algebra is associative. Notice also that B is linear in D (*i.e.* of the correct order), and is hermitian for hermitian $A_{(1)}$. Finally, it is suggested that the reader check the symmetry of equation (3.29) for the almost-associative geometry that we give in section 4.

4. The Simplest Non-Associative Example

Our goal in this section is to construct the simplest nonassociative spectral triple, and compute the corresponding spectral action explicitly.

4.1 The spectral triple

We will consider the finite nonassociative spectral triple given by

$$F = \{\mathcal{A}_F, \mathcal{H}_F, D_F, \gamma_F, J_F\} = \{\mathbb{O}, \mathbb{O}, 0, \mathbb{I}, J_{\mathbb{O}}\} \quad (4.1)$$

where $J_{\mathbb{O}}$ denotes octonionic conjugation. The $K0$ dimension of this spectral triple is zero. Note that, once we choose $\gamma_F = \mathbb{I}$, the choice $D_F = 0$ is forced upon us by the requirement that $\{D_F, \gamma_F\} = 0$ (see subsection 2.2). Because $D_F = 0$, when we calculate the corresponding spectral action below, we will obtain a model in which the fermion fields are massless and there are no Higgs fields. Again, this is a consequence of the fact that we are considering the simplest nonassociative spectral triple. In a forthcoming paper we will show how to construct realistic models with Higgs fields, spontaneous symmetry breaking and fermion masses [25].

We would like to stress that we can arrive at the nonassociative spectral triple F , and check that it makes sense, in two different ways. On the one hand, F satisfies all of the required axioms for a spectral triple (including the appropriate nonassociative generalizations of the order zero and order one conditions presented in Subsection 3.3), and is compatible with the principle of automorphism covariance, as explained in Subsection 3.2. On the other hand, we can start with the associative spectral triple previously presented as “example 2” in Subsection 2.2: $F_0 = \{\mathbb{K}G, \mathbb{K}G, 0, \mathbb{I}, J_{\mathbb{K}G}\}$, where $\mathbb{K}G$ is the group algebra based on $\mathbb{K} = \mathbb{R}$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and $J_{\mathbb{K}G}$ denotes the natural $*$ operation in $\mathbb{K}G$ (see Subsection 2.2). As already noted in Subsection 2.2, F_0 satisfies the standard axioms for an associative spectral triple of $K0$ dimension zero. But then, when one twists $\mathbb{K}G$ into \mathbb{O} (see [29]), the associative spectral triple F_0 is correspondingly twisted into our nonassociative triple F .

Next we construct the “almost-associative geometry” $M \times F$: *i.e.* the product of the canonical spectral triple M with the finite-dimensional nonassociative triple F (see

Subsection 2.2):

$$M \times F = \{C_\infty(\mathcal{M}, \mathbb{O}), L^2(\mathcal{M}, S) \otimes \mathbb{O}, D_c \otimes \mathbb{I}, \gamma_c \otimes \mathbb{I}, J_c \otimes J_\mathbb{O}\}. \quad (4.2)$$

Note that \mathbb{O} is an algebra over \mathbb{R} , so we take the tensor product $C_\infty(\mathcal{M}, \mathbb{R}) \otimes \mathbb{O}$ over \mathbb{R} , as in [34, 35], to obtain $C_\infty(\mathcal{M}, \mathbb{O})$, the algebra of smooth functions from \mathcal{M} to \mathbb{O} .

4.2 Fluctuating D for the Octonionic example

The first task in constructing the spectral action for our almost-associative geometry is to fluctuate the Dirac operator $D = D_c \otimes \mathbb{I}$. Before doing so however it will be useful to discuss the automorphism group corresponding to our $*$ -algebra $\mathcal{A} = C_\infty(\mathcal{M}, \mathbb{O})$. Start by considering the automorphisms of the octonions themselves \mathbb{O} . If we choose any two elements $a, b \in \mathbb{O}$, then the operator

$$\delta_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b] \quad (4.3)$$

acts as a derivation on \mathbb{O} ; indeed, we can write any derivation δ on \mathbb{O} as a sum of $\delta_{a,b}$'s:

$$\delta = \sum_{a,b} \delta_{a,b} \quad (4.4a)$$

for some choice of a 's and b 's. In fact, the space of derivations of \mathbb{O} is only 14 dimensional. To see this, choose a standard basis on \mathbb{O} , consisting of the unit element e_0 , along with 7 orthonormal anti-hermitian elements $e_i^* = -e_i$ ($i = 1, \dots, 7$). Then we can expand any derivation δ as $\delta = \sum_{i,j=1}^7 c^{ij} \delta_{e_i, e_j}$, where the coefficients c^{ij} are real numbers. Furthermore, the derivations δ_{e_i, e_j} are not all linearly independent: first, they satisfy $\delta_{e_i, e_j} = -\delta_{e_j, e_i}$ (which cuts their number down to 21); and second, they satisfy the Jacobi-like identity $\delta_{[e_i, e_j], e_k} + \delta_{[e_j, e_k], e_i} + \delta_{[e_k, e_i], e_j} = 0$, which gives 7 more constraints and cuts the number from 21 to 14. Denote the 14 independent derivations by δ_A ($A = 1, \dots, 14$). Then we can expand any derivation δ as

$$\delta = \sum_{A=1}^{14} c^A \delta_A \quad (c^A \in \mathbb{R}). \quad (4.4b)$$

The 14 independent generators δ_A form a 14-dimensional Lie algebra: namely, the smallest exceptional Lie algebra, \mathfrak{g}_2 .

Now the automorphism group of the $*$ -algebra $C_\infty(\mathcal{M}, \mathbb{O})$ is the semi-direct product of two pieces: the inner automorphisms and the outer automorphisms. Consider an element $f \in C_\infty(\mathcal{M}, \mathbb{O})$ – i.e. a smooth function f from \mathcal{M} to \mathbb{O} . The outer automorphisms act as $f(x) \rightarrow f(\phi(x))$, where ϕ is a diffeomorphism of \mathcal{M} , while the inner

automorphisms send $f(x) \rightarrow e^{\delta(x)} f(x)$. Here we can again write $\delta(x)$ in two different ways. On the one hand, we can write

$$\delta(x) = \sum_{a(x), b(x)} \delta_{a(x), b(x)} \quad (4.5a)$$

for some choice of $a(x)$'s and $b(x)$'s; this expansion is analogous to Eq. (4.4a), except the a 's and b 's are now drawn from $C_\infty(\mathcal{M}, \mathbb{O})$ rather than \mathbb{O} . On the other hand, we can write

$$\delta(x) = \sum_A c^A(x) \delta_A \quad (4.5b)$$

where δ_A are the 14 generators of G_2 introduced above; this expansion is analogous to Eq. (4.4b), except the coefficients $c^A(x)$ are real functions rather than real numbers. When we represent the algebra, the corresponding representation of this derivation is given by $\tilde{\delta}(x) = \sum_{\tilde{a}(x), \tilde{b}(x)} \delta_{\tilde{a}(x), \tilde{b}(x)} = \sum_A c^A(x) \tilde{\delta}_A$.

Let us now fluctuate the Dirac operator to account for the inner automorphisms. Starting with the un-fluctuated Dirac operator $D = -i\gamma^\mu \partial_\mu \otimes \mathbb{I}$ we have, following section 3.4:

$$D \rightarrow D' = e^{\tilde{\delta}(x)} D e^{-\tilde{\delta}(x)} = -i\gamma^\mu \otimes (\partial_\mu + e^{\tilde{\delta}(x)} [\partial_\mu, e^{-\tilde{\delta}(x)}]). \quad (4.6)$$

To first order, and following the notation in equation (4.5a), D' can be written as

$$\begin{aligned} D' &\simeq -i\gamma^\mu \otimes (\partial_\mu - [\partial_\mu, \tilde{\delta}(x)]) \\ &= -i\gamma^\mu \otimes (\partial_\mu - \sum_{\tilde{a}(x), \tilde{b}(x)} (\delta_{\partial_\mu \tilde{a}(x), \tilde{b}(x)} + \delta_{\tilde{a}(x), \partial_\mu \tilde{b}(x)})). \end{aligned} \quad (4.7a)$$

Or, using the notation given in (4.5b)

$$D' \simeq -i\gamma^\mu \otimes (\partial_\mu - [\partial_\mu, c(x)^A] \tilde{\delta}_A) \quad (4.7b)$$

The inhomogeneous terms in equations (4.7a) and (4.7b) tell us the form that our general fluctuations should take. The Dirac operator with inner fluctuation terms is then given by

$$D_I = -i\gamma^\mu \otimes (\partial_\mu + B_\mu), \quad (4.8)$$

where

$$B_\mu = \sum [A_\mu, A_{(0)}] - [A_\mu, J A_{(0)} J^*] + J[A_\mu, A_{(0)}] J^* := B(x)_\mu^A \tilde{\delta}_A, \quad (4.9)$$

and the one forms are given as $A_\mu = \sum L_{a(\tilde{x}) \partial_\mu b(\tilde{x})}$, and $B(x)_\mu^A$ is a general spatially dependent gauge field. So far we have only accounted for inner fluctuations. The fully fluctuated Dirac operator must also account for outer automorphisms of the algebra. It is given by $D_* = -i\gamma^\mu \nabla_\mu^E$ where $\nabla_\mu^E = \nabla_\mu^S \otimes \mathbb{I} + \mathbb{I}_4 \otimes B_\mu$, and ∇_μ^S is the usual Levi-Civita connection on the spinor bundle [15].

4.3 Calculating the action

Having determined the form of the full fluctuated dirac operator, we can now perform the heat kernel expansion for the spectral action. For an explanation of how to perform this expansion, see Ref. [15]. The spectral action is given by:

$$S_b = Tr \left(f\left(\frac{D_*}{\Lambda}\right) \right) \quad (4.10)$$

where f is a real, even function. Before we can perform the heat kernel expansion we first need to calculate the square of the fluctuated dirac operator, which is given by

$$\begin{aligned} D_*^2 &= (-i\gamma^\mu \nabla_\mu^S \otimes \mathbb{I} - i\gamma^\mu \otimes B_\mu)^2 \\ &= \Delta_A^E - \frac{1}{2}\gamma^\mu \gamma^\nu \otimes F_{\mu\nu} - \frac{1}{4}R \otimes \mathbb{I}, \end{aligned} \quad (4.11)$$

where R is the Ricci scalar, and

$$\Delta_A^E = -g^{\mu\nu} \nabla_\mu^E \nabla_\nu^E \quad (4.12)$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]. \quad (4.13)$$

Equation (4.10) can then be expanded as

$$Tr \left(f\left(\frac{D_*}{\Lambda}\right) \right) = 2f_4\Lambda^4 a_0(D_*^2) + 2f_2\Lambda^2 a_2(D_*^2) + f(0)a_4(D_*^2) + O(\Lambda^{-1}) \quad (4.14)$$

where the $f_n = \int_0^\infty f(x)x^{n-1}dx$ ($j > 0$) and $a_k(D_*^2)$ are the Seeley-deWitt coefficients. For a compact Euclidean manifold without boundary we have

$$a_0(D_*^2) = \int_M \frac{1}{4\pi^2} \sqrt{g} d^4x \quad (4.15)$$

$$a_2(D_*^2) = \int_M \frac{R}{48\pi^2} \sqrt{g} d^4x \quad (4.16)$$

$$\begin{aligned} a_4(D_*^2) &= \int_M \frac{1}{16\pi^2} \frac{1}{360} Tr \left[\left(\frac{5}{4}R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right. \right. \\ &\quad \left. \left. + 45\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma F_{\mu\nu}F_{\rho\sigma} + 30\Omega_{\mu\nu}^E (\Omega^E)^{\mu\nu} \right] \sqrt{g} d^4x, \end{aligned} \quad (4.17)$$

where $\Omega_{\mu\nu}^E = \Omega_{\mu\nu}^S \otimes \mathbb{I} + \mathbb{I}_4 \otimes F_{\mu\nu}$, and $Tr(\Omega_{\mu\nu}^S \Omega^{S\mu\nu}) = -\frac{1}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. The full bosonic action is then

$$\begin{aligned} S_b &\simeq \int_M \frac{N}{(4\pi)^2} \sqrt{g} [8f_4\Lambda^4 + \frac{2}{3}Rf_2\Lambda^2 \\ &\quad + f(0)\frac{N}{360}(5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{240}{N}Tr(F_{\mu\nu}F^{\mu\nu}))] \end{aligned} \quad (4.18)$$

Finally, the fermionic action is given by

$$S_f = \langle \psi | D_* | \psi \rangle = \int \psi_i^\dagger(x) \mathcal{D}^{ij} \psi_j(x) \sqrt{g} d^4x, \quad (4.19)$$

where $\mathcal{D}^{ij} = -i\gamma^\mu(\nabla_\mu^s \delta^{ij} + B_\mu^A \tilde{\delta}_A^{ij})$, is hermitian. (Note that, in this equation, the first δ_{ij} denotes an ordinary Kronecker delta, not a derivation.) In our convention the generators $\tilde{\delta}_A^{ij}$ are anti-hermitian, which means that B_μ^A is hermitian.

The full action of our theory is given by the sum of both the Bosonic action and the Fermionic action $S = S_b + S_f$. It describes Einstein gravity coupled to a G_2 gauge theory, with 8 massless Dirac fermions which split into a singlet and a septuplet under G_2 . In this paper, we have just presented the simplest possible model by way of illustration. In a forthcoming paper [25], we show how to construct realistic models that include Higgs fields, spontaneous symmetry breaking and fermion masses.

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